Ave Maria, Fractals & Mathematics

by **Marcio Luis Ferreira Nascimento**

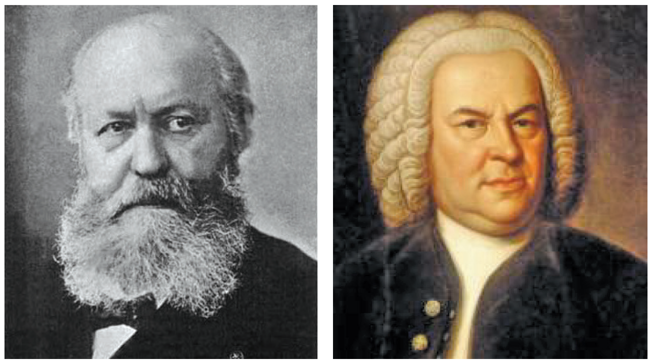
Music critics have compared some *classical* music like that from Bach to the precision of mathematics. Mathematicians disagree on a precise definition, but a fractal is typically described as exhibiting self-similarity, which means identical (or nearly identical) patterns appear, whether the shape is viewed from up close or far away. That is, the part looks like the whole, and the whole looks like a part. However, identifying fractals in music requires a different approach than seeing them in an image. The purpose of this short note is to suggest that Ave Maria of Bach/Gounod could be related to mathematics, at least in part (or just *viewed*), as a sound example of Mandelbrot's fractal geometry, due to its simple *self-similarity* on melody.

As you know, music can be used to express human feelings and emotions. Unlike most places in the world, at 18:00 there is a beautiful time for listening to the radios in the city of San Salvador, Bahia, Brazil. It is possible to hear the beautiful 'Ave Maria' (Gounod, 1871) one of the world's most famous compositions, recorded on the Latin text of the prayer with the same name. The piece was developed in 1853 from a melody of romantic French composer Charles Gounod (1818–1893) (Fig. 1), inspired and specially designed to be superimposed on Prelude No. 1 in C major, BWV 846, from Book I of 'The Well-Tempered Clavier' (in German, *Das Wohltemperierte Klavier*) (Bach, 1722). In fact, Gounod had published its composition under the title: '*Meditation sur le Premier Prélude de Piano de S. Bach*' (something like 'Meditation on the First Piano Prelude of S. Bach').

The composer of such a magnificent masterpiece that inspired Gounod was none other than the incomparable German composer Johann Sebastian Bach (1685–1750) (Fig. 1). Bach was above all a universal genius: singer, harpsichordist (pianist), conductor, organist, teacher, violinist and violist. This work is considered one of the landmarks in the history of European music, one of the most important musical works of the Occident, with large-scale, depth, diversity, fine aesthetics and enormous complexity. He gave the title to a book of preludes and fugues in all 24 major and minor keys, composed 'for the profit and use of musical youth desirous of learning, and especially for the pastime of those already skilled in this study'.

'Ave Maria', also called the '*Angelical Salutation*', has an interesting musical structure. His main theme (or core) is recurrently repeated, repeated and repeated, but not the same way for a common listener. Such a structure shaped sound has similarities with diagrams and visual schemes that mathematicians call *fractals*.

The term comes from Latin '*fractus*' and means broken,



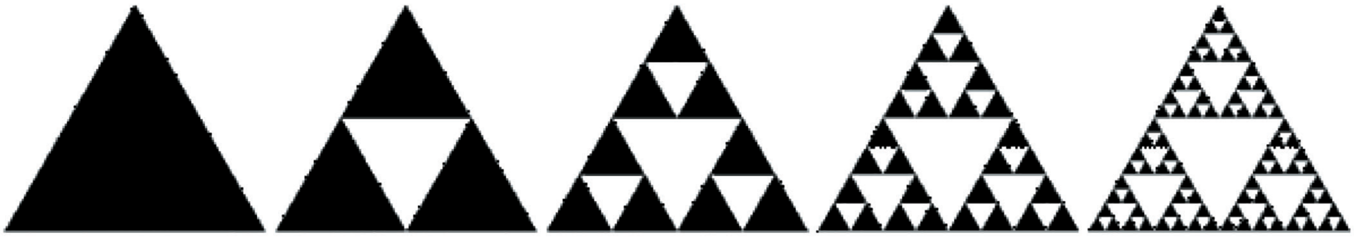
Charles Gounod (1818–1893) Johann Sebastian Bach (1685–1750)

Fig. 1 The composers Gounod and Bach

irregular, discontinuous. It was defined by the French mathematician Benoît B. Mandelbrot (1924–2010) in his amazing book 'The Fractal Geometry of Nature' in 1982 (Mandelbrot, 1982). One of the most beautiful and simple fractals can be easily done using paper and pencil. It is based on an equilateral triangle (i.e. three equal sides), and first described by Polish mathematician Waclaw Franciszek Sierpinski (1882–1969) in a scientific paper in 1916 (Sierpinski, 1916), which is obtained as the limit of a recursive process. To begin, we start from a simple triangle (Fig. 2, stage 0). Thereafter joining the midpoints of each edge of the triangle into four new smaller triangles (Fig. 2, stage 1). Disregard the lower and central triangle, just consider only three of the remaining triangles. This recursion is repeated, over and over for each new design obtained.

Notice that the resulting object, known as the Sierpinski Triangle, looks exactly like any one of the three sub-triangles that comprise it, except for scaling. It also looks exactly like any one of the nine filled-in sub-subtriangles at the next stage, and so on. The construction guarantees this self-similarity because the process of replacement is identical for each sub-triangle, as it was for the whole triangle.

Another way, or if you want another recipe ('algorithm'), to create this same fractal is called the 'collage method'. Again consider an initial triangle (Fig. 2, stage 0), draw a similar one and make three smaller copies, each half



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4

Fig. 2 Sierpinski triangle construction, from initial stage (0) up to the 4th stage

as tall and half as wide as the original, (Fig. 2, stage 1). Repeat the process for each of the three copies, again reducing by half the size for each new operation. This is one of the most important properties of the fractal: *self-similarity*, which basically means a similarity at any scale, which means that some part may represent a portion of the whole. Another interesting property involves a fractional dimension, which is obtained through the number of copies and the scale factor.

But what is the Sierpinski Triangle dimension? A straight line is 1-dimensional. A filled-in triangle is 2-dimensional. Somehow we intuitively know these facts. A solid cube is 3-dimensional. Thus, for integer number of dimensions d, when doubling a side of an object, 2^d copies of it are created, i.e. 2 copies for 1-dimensional object, 4 copies for 2-dimensional object and 8 copies for 3-dimensional object. Notice that three copies of the Sierpinski Triangle can be assembled to create a larger version and that the larger version is twice the size of the original one (the scaling factor in this case is 2). Thus, the dimension d of the Sierpinski Triangle is the number such that $2^d = 3$. We can work out this dimension using a calculator by computing $d = (\ln 3)/(\ln 2) \cong (0.47714)/(0.30103) \cong 1.58496...$ So, we see that the dimension of this fractal is greater than 1 but smaller than 2 (Burger and Starbird, 2009). With these interesting properties, fractals may form figures with a virtually infinite amount of details.

Returning to music, there are several different instrumental arrangements for 'Ave Maria', including for violin and guitar, string quartet, harpsichord (obviously, due to the origin of the song), solo piano, cello, trombones and even a cavaquinho (a Brazilian small string instrument of the European guitar family) version. Over the centuries, the Bach Ave Maria Gounod has been sung in many different styles and languages, including by Céline Dion, Beyoncé, Alessandro Moreschi, Maria Callas, Luciano Pavarotti, Sarah Brightman, Jose Carreras, Helene Fischer, Nina Hagen, Andrea Bocelli, Karen Carpenter, Leona Lewis, Sandi Patty and Arielle Dombasle. Curiously, it is the repetition of a particular theme throughout all the song that mathematicians understand by *self-similarity*, repeating, repeating and repeating ... like a kind of fractal sound. It is important to note that there are other fractal aspects for 'Ave Maria' and other classical music, but they are a bit far beyond the scope of this text (Hsu, 1990; Su and Wu, 2006; Ornes, 2014; Voss and Clarke 1975).

The most curious result is that fractals can describe the world as it is. As said in the introduction to Mandelbrot's brilliant book: 'clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line'. Whirlpools, the dripping of a tap, the heartbeat when we love, the heavenly bodies dance, the branching of blood vessels, changes in the weather or financial markets, even the 'Ave Maria' song... can be seen as fractal representations, the true geometry of nature. *Amen*.

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